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ABSTRACT

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ANALYZING PROBLEM SOLVING PROCESSES INVOLVED WITH PROBLEMS FROM INCIDENCE GEOMETRY - AN EXPLORATORY STUDY

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Abstract

A three-dimensional classification system of mathematical problems is given.A background philosophy was constructed through an analysis of different approaches to problem solving research, leading to the statement of five theses and the objectives for the study.

An exploratory study of mathematical problem solving processes was conducted within this framework. Think-aloud sessions of students were vidiotaped while solving problems from incidence geometry. A category system was developed to form data from the observations. Correlation-methods and sequence-analysis techniques were used to analyze the data. Hypotheses were generated concerning systematic behavior (strategies), problem solving success, modes of representation, mistakes and problem solving types.

Analyzing problem solving processes involved with problems from incidence geometry - an exploratory study(*)

Bernd Zimmermann

1. The framework of the study

1.1 Preleminaries

The starting-point of this section can be summarized by the following statement of Bauersfeld (1977):
"The outcomes of an investigation are always determined in advance by the fundamental theories on which hyotheses have been developed." (cf. the statement of Einstein (Heisenberg (1969)): "Just the theory decides, what is observable." and similar statements of Feyerabend (1975)).

Accepting these statements, the following conclusions could be drawn:

- In any kind of research (at least in mathematics education) the underlying theories should be made conscious, because very often different results about the same topic could be explained by different "hidden" theories of different researchers.
- The following report should be an example of such an attempt of making a underlying theory explicit.
- Additionaly a possible direction of problem solving research should be set forth by the following theory (the framework of the study).

(*) This is an abstract of a doctoral thesis, the author carried out at the University of Paderborn. The project was sponsored by the Minister of Science and Research of Northrhine-Westfalia.

1.2 Classification system for mathematical problems

The study of problem solving should clarify the concept of a "problem" given in a mathematics lesson.

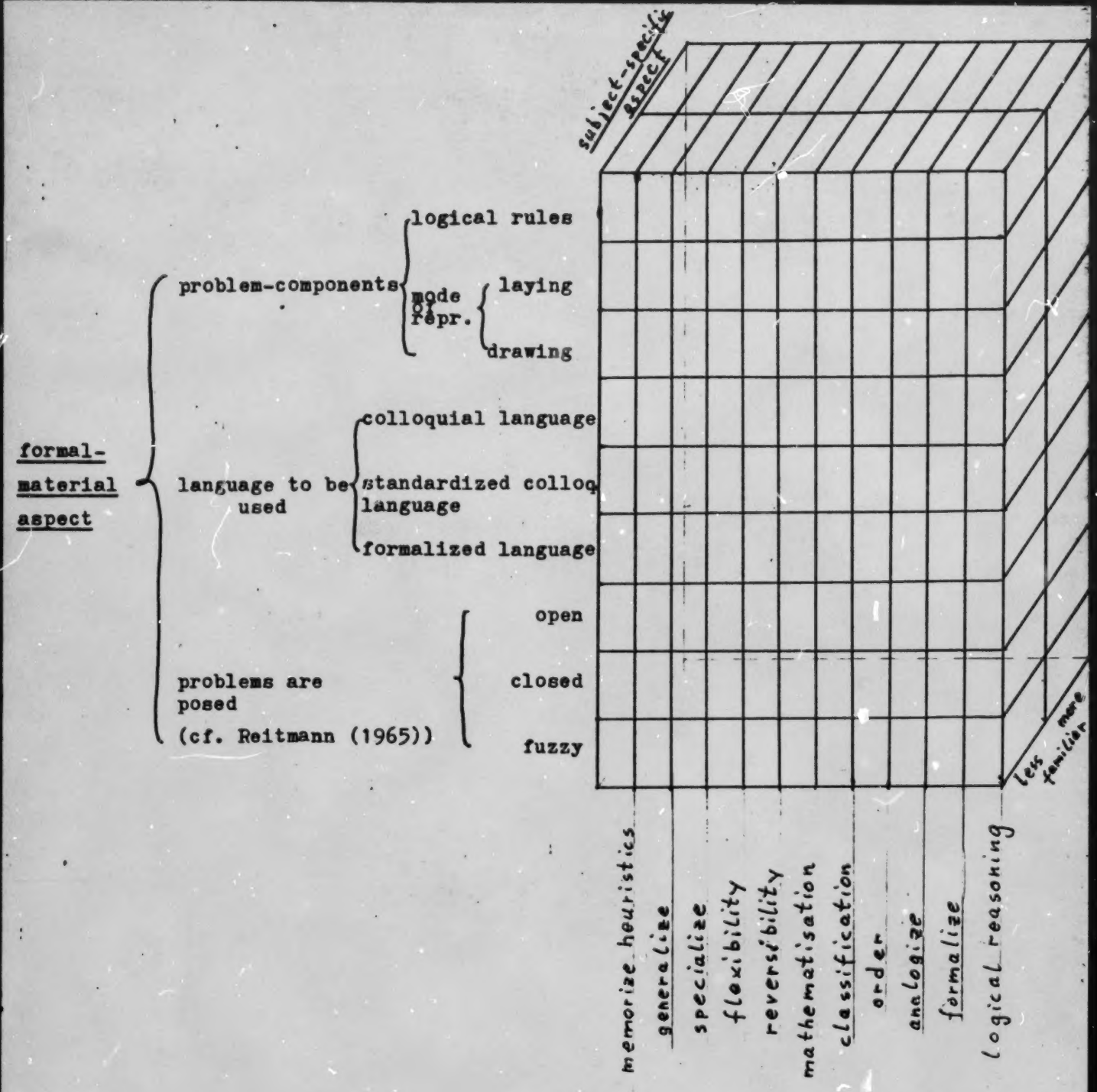
At least three aspects seem to be important:

- the formal-material (What is the problem ?)
- the subject-specific (To whom is the problem given ?)
- the norm-oriented (For what objective is the problem given ?)

These aspects are not independent since the norm-oriented aspect (e. g. the objectives in mathematics education) has a strong influence on the other ones (cf. the above mentioned quotations of Bauersfeld and Einstein !).

These three aspects established the basis for a three-dimensional classification system for mathematical problems. One of the objectives of this study is to give a possible domain of generalizability of problem soving observations and instructions for each "cell" of this "cube". The essential structure of the system is represented in Fig. 1. Examples about its operation will be given later.

Fig. 1



Classification system for mathematical problems

Fig. 1

Concentration upon problem solving processes, which belong to special cells or set of cells of this cube, seems to be more promising than searching for general heuristics, because of the fact that e. g. Webb's (1975) observations support the presumption, that most heuristics are problem-specific.

The following report will be a description of the generation of the objectives of the study by marking three stages:

- At first leading ideas are presented, which establish the background-philosophy for the following analysis of different approaches to problem solving research (1.3).
- This analysis is carried out and closed by five theses, which summarize the results of the preceding discussion, stressing mainly additional methodological aspects (1.4).
- The chapter is closed by the final objectives of the study (1.5).

1.3 Background-philosophy

The leading ideas behind this study are as followed:

a) Process before procuct

More stress should be put on the analysis of the problem solving process than on the product (cf. e.g. Kantowski 1977, 1978).

b) Subject-centration

One of the most important aims in problem solving research should be the description of problem solving processes which takes into account the subject's own verbal interpretation of these processes. This judgement is supported by the following statement of Bruner (1964):

" If information is to be used effectively, it must be translated into the learner's way of attempting a problem. If such translatability is not present, then the information is simply

useless." It will be demonstrated later that this aspect seems to be neglected to that moment. This idea could be found in the "action-research" concept which had been founded by K. Lewin (1953)

c) More stress on poor problem solving

Not only good (cf. Polya 1957, Wickelgren 1974) but more strongly poor problem solving (cf. Hadamard 1949, Radatz 1979) should be explored. What is meant by "good" and "poor" should be made explicit and conscious in any case. E. g. a teacher may put more stress on correctness and completeness of a problem solving process than on the mere representation of fundamental solution-ideas or vice versa.

Again Lakatos (1976, 1978) demonstrated that an error even in mathematics may depend on "hidden" norms and axioms.

So error analysis has not only a psychological but also a normative aspect.

d) Individual differences

More stress should be put on the research of problem solving styles (cf. e. g. Kilpatrick 1967, Webb 1975, Radatz 1976).

e) Operationalization

"The language used to describe problem solving processes is cumbersome. The meaning of strategy, heuristic or heuristic strategy varies from study to study."(Webb 1975)

So a more extensive operationalization might cancel misunderstandings and those difficulties, which are connected with an effective application of instruction-programs in problem solving (cf. Fischbein 1973).

f) Plurality

Especially exploratory studies should be concerned with a lot of relevant (and seemingly irrelevant) aspects of problem solving behavior. "But in an exploratory study, where payoffs are difficult to estimate, it seems better to err in the direction of generosity than of parsimony." (Kilpatrick 1967)

1.4 Classical and new research on problem solving processes

1.4.1 Classical research

Mathematicians (i. e. Polya) and gestalt psychologists (i. e. Wertheimer)are mainly interested in the description of good and ideal problem solving behavior. They do not care so much about bad problem solving, as rather often observable "in nature". These authors give a lot of hints to become a good problem solver. But there is less advice what to do if a pupil has difficulties or makes mistakes.

Especially the terminology of gestalt psychologists seems to be sometimes a little bit fuzzy so that sharpening would be useful. Mcreover this approach suffers from a lack of "subject-centration".

A lot of information about individual differences with respect to mathematical abilities and other personal structures is given by the factor analytic approach (cf. e. g. Radatz 1976). Unfortunately it is not possible to get some idea about the sequential or causal organization of problem solving processes in this way (cf. the corresponding statement of Webb (1975)). Scandura (cf. Kilpatrick (1965)) asserted that such information could be useful for the right timing of problem solving advice.

In contrast to the gestalt approach, the behavioristic approach puts more emphasis on precise description and operationalization of problem solving behavior. Some well-known representatives are Suppes (1969) and Dienes & Jeeves (1971, 1972).
Nevertheless, by emphasizing observable behavior the "natural" situation rather often suffers from a severe deformation and the area of research from a considerable restriction. In addition there is a lack of "subject-centration" like in the preceding approaches.

The information-processing-approach (cf. Newell & Simon (1972)) represents most accurately the fundamental experience, that problem solving is a complex process.
One of the disadvantages of this approach is the fact that its terminology is oriented more to the internal life of a computer than to that of a subject. Moreover, questions concerning motivation or individual differences are no topic of this approach though surely important for school-reality (and so for mathematics education).

It is possible to characterize the Soviet approach to problem solving by the following statement of Rubinstein (1961):
"The fundamental form of a psychic being is its existence as a process." In addition to this reasonable assumption the orientation towards school-reality seems to be very useful.
Thus Soviet researchers also make a lot of classroom-observations. But because of rather uncontrolled boundary-conditions it seems to be as difficult to generalize the results of such research as the results of the analysis of controlled single-sessions with classroom-problems. - Checking e. g. the outstanding work of Krutetskii (1976), one can conclude that the understanding of the term "process" in Soviet research is rather similar to that of the terms "variable" or "factor" in the factoranalytic approach. So there is in general no examination of sequential connections between the single variables. Moreover the behavior of a problem-solver is rather often interpreted with respect to a very rigid frame, so that there is no place for "subject-centration", the self-understanding of the subject.
The approach which has to do with standardized classroom-observation by a fixed category system (e. g. that of Flanders (1962)), seems to be the best one, because it is the most praxis-oriented one. Nevertheless, information-gathering is rather difficult:

A coding by the category system of Flanders can only describe the top of an iceberg. If one tries (like e. g. Jüngst (1977) did) to get some more information about the problem solving process by a pupil-protocol, a rather different process is generated than the one, which one originaly wanted to observe. - So there are also a lot of difficulties concerning generalisation, similar to the situation with data from single-sessions.

1.4.2 New research on problem solving processes.

Three modern lines are presented, which can be taken mainly as consequences of the foregoing approaches (only the main representatives are quoted):

a) The Polya-school (Kilpatrick (1967), Webb (1975)).

b) Synthesis of American and Soviet approaches (Kantowski (1974, 1977)).

c) Further developments of the information-processing approach in Western Europe (Dörner (1974), Lüer (1973)).

ad a): This important branch got a very strong impulse ten years ago by the doctoral thesis of Kilpatrick (continued e. g. by the study of Webb): He studied problem solving behavior with the aid of a category system, which is related to the terminology of Polya. His results stress the importance of Polya's suggestions for instruction. - Nevertheless, it would be usefull to intensify sequential analysis of problem solving processes. An additional subject-centration of the category system seems to be necessary.

ad b): The previous comment pertaines also to Kantowski's thesis. She is one of the representatives of the Soviet-American approach to problem solving: she tries to synthesize American research design with Soviet praxis-orientation.

ad c): Lüer can be taken as one of the representatives of the further developed information-processing approach in Western Europe. With the aid of "sequence-analysis" he discovered a "Grundrhythmus" (fundamental rhythm) which could describe the problem solving process for a special class of logical problems.

(cf. Newell & Simon (1972) and Whitehead & Russel (1910)).

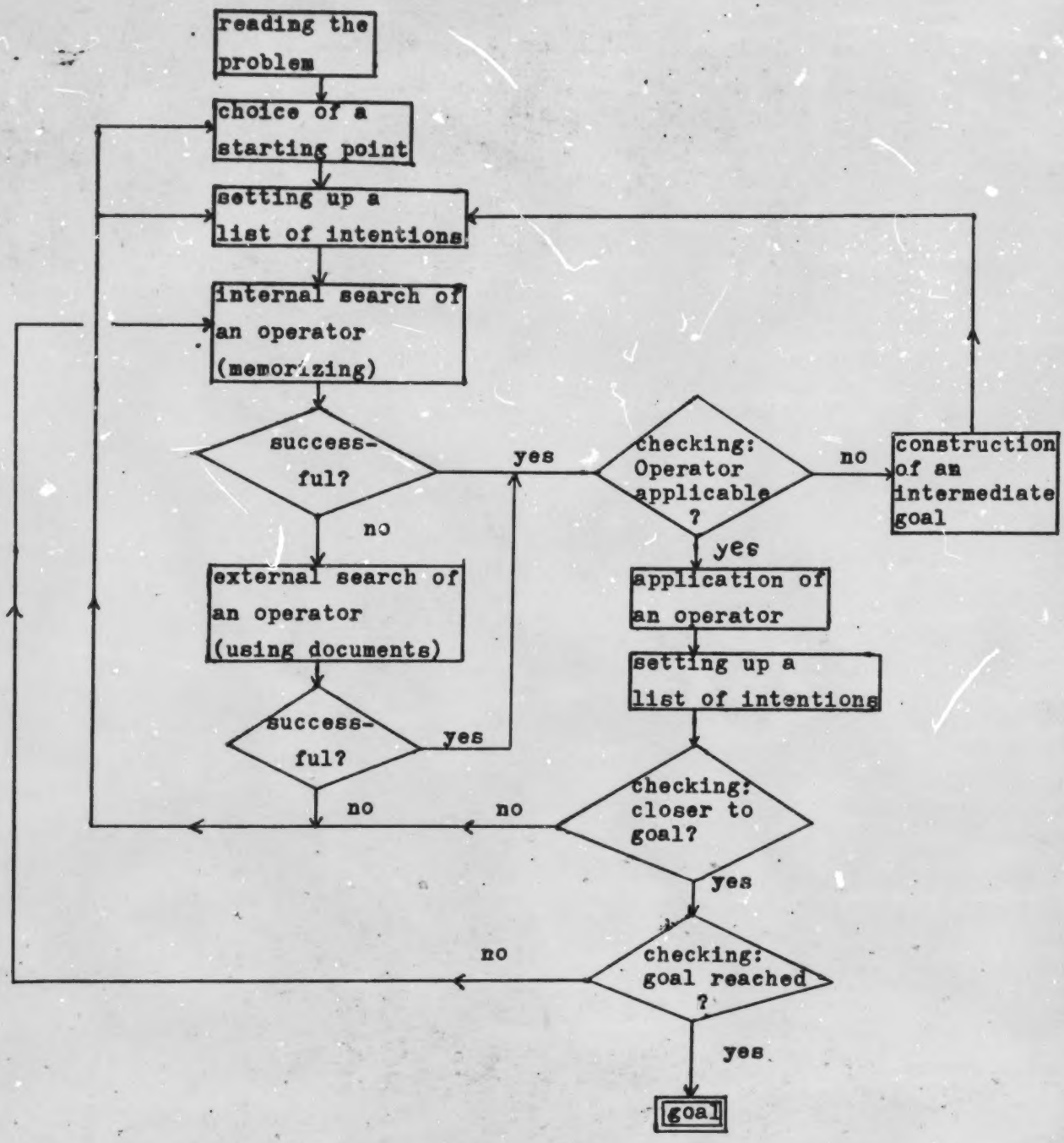
Example: given: Axioms, rules for deduction,

$\neg (p \wedge q)$

wanted: $\neg p \vee \neg q$.

The "fundamental rhythm" found by Lüer and Dörner is presented in Fig. 2.

Fig. 2



Fundamental rhythm involved with problems from Aristotelian logic (Dörner/Lüer)

Fig. 2

Though there is a strong stress of operationalization and process-analysis, Lüer and Dörner do not care about individual differences and subject-centration of their category system. Henceforth the "think-aloud" method, which they apply like most of the other researchers, should be more standardized.

1.5 Consequences - methodological framework

The foregoing analysis can be summarized in the following way:

- No approach is explicitly oriented towards the terminology and self-understanding of the observed subject.
- It seems best to combine the factoranalytic and the information-processing approach. In this way the specific advantages of each approach (namely process-description, respectively individual differences) could cancel their respective disadvantages (namely no individual differences, no process description).

As a consequence a methodological framework for the exploratory analysis of mathematical problem solving processes is suggested by stating the following five theses:

Thesis 1: It is important for a useful description of problem-solving behavior to get some information about the "subjective" problem solving processes. They should be brought into connection with "objective" problem solving processes. By the last term processes are meant, which are taken by the researcher to be useful or important (e. g. Polya's heuristics).

Thesis 2: It is possible and useful to construct a category system of fundamental activities which are related to the self-understanding of the subject and the observer (researcher). It is possible to describe and analyze problem solving as a sequence of such fundamental activities.

Thesis 3: Problem solving processes are observable and analyzable in single sessions.

Thesis 4: The "think-aloud" method - connected with minimized and standardized influence of the interviewer - fits best to get information especially about the subjective representation of problem solving processes.

Thesis 5: Hypotheses about possible global connections of fundamental activities can be obtained by correlation-methods. Hypotheses about possible local connections are available by "sequence-analysis".

1.6 The objectives of the study

1. A category-system was developed which would enable the description of all recognizable information in a standardized way. It was especially important to recognize not only the point of view of the evaluator ("objective aspect") but also of the subject ("subjective aspect").
2. The method of solving problems from incidence geometry was compared with the "fundamental rhythm" found by Lüer (1973) and Dörner (1974) in analyzing think-aloud protocols from logical problems.
3. Local and global conditions were analyzed for successful and non-successful problem solving.
4. The connection between the use of different modes of representation and problem solving success was checked.
5. The use of different modes of representation was checked to determine different problem solving styles.
6. Typical mistakes made by students solving problems from incidence geometry were determined.

2.The study

2.1 Subjects

Twenty students volunteered for the study (10 male, 10 female, average age 21). Among these were 16 freshmen, who majored mathematics education at the University of Paderborn. 4 subjects were higher grade students from a secondary school in Paderborn. None of the subjects was enrolled in a special geometry course before. The 16 university students were average-ability students, as it was demonstrated by the result of written exercises within their normal university-educational program in mathematics education. The secondary school students turned out to be as capable as the university students by the subsequent experience.

2.2 Problems

The problems were selected from incidence geometry.
The following system of axioms was given to the subjects (abbreviated version):

(A1) To any two different points there exists exactly one connecting curve.

(A2) To any curve and any point there exists exactly one parallel curve, meeting this point.

(A3) There exist at least three noncolinear points.

After a short introcuction into incidence geometry by means of written material, including the axioms and some examples for their application, the subjects had to prove the following theorems:

Theorem 1: Supposition: Let k be an arbitrary curve

Proposition: Then exists always a curve with the following properties:

(i) $l \parallel k$

and

(ii) $l \not\parallel k$.

Theorem 2: Supposition: Let k, l, s be any curves with the following properties:

(i) k ‖ l

and

(ii) l ‖ s.

Proposition: Then k ‖ s is valid, too.

Theorem 3: Supposition: no additional (except axioms and already proved theorems)

Proposition: There exist (at least) 4 points with the following properties:

(i) they are all distinct

and

(ii) no three of them are lying on the same curve.

The original version of the theorem had been a more abbreviated one (Th. 1:"To any line exists a distinct parallel line " etc., cf. the axioms above). But because of experience resulting from a pre-test made with five other subjects of similar age and qualification, the text of the theorems (and axioms, too) was presented in greater detail to the subjects of the main-study to make details clearer. Especially the difference between the supposition and proposition of the theorems was made more explicit in this way. The term "curve" turned out to be more suitable than "line" (giving more flexibility).

The subjects had to solve two further problems which were not incorporated in the evaluation process because of the considerable amount of datas already derived from the first three problems.

The reasons for choosing the problems from incidence geometry are the following:

a) The problems should be similar to the (axiomatic) structure of the problems from Aristotelian logic used by LÜer (1973) and Dörner (1974) (cf. Fig. 3, 4), because a comparison was planned with their studies.

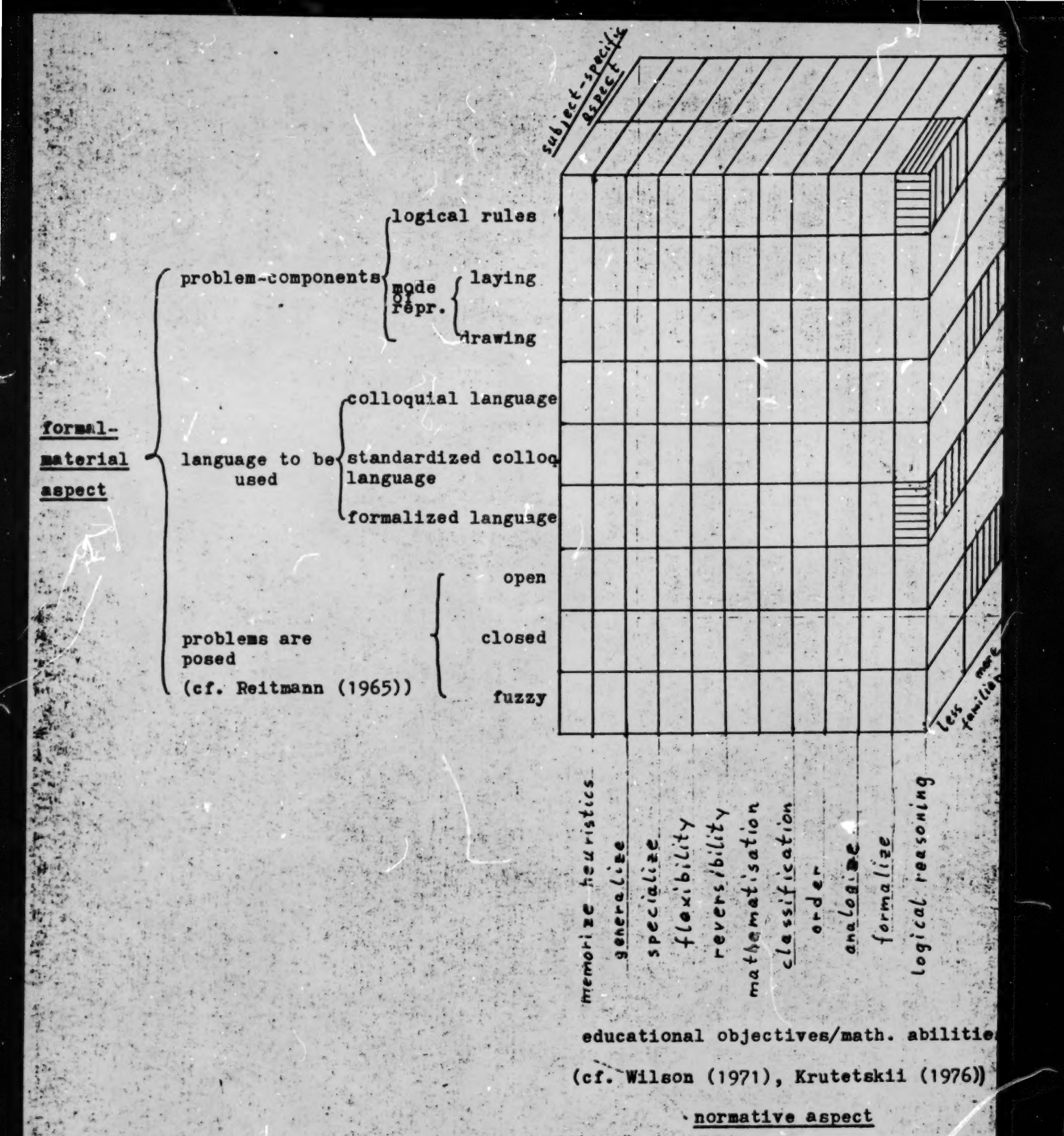
b) The problems should incorporate different possibilities to visualize, because the import of the different modes of representation (cf. Bruner (1964)) in the problem solving process should be analyzed.

c) There was no need for an extensive instruction.

d) Axiomatic geometry is one of the topics in secondary schools and for students in mathematics education in Western Germany.

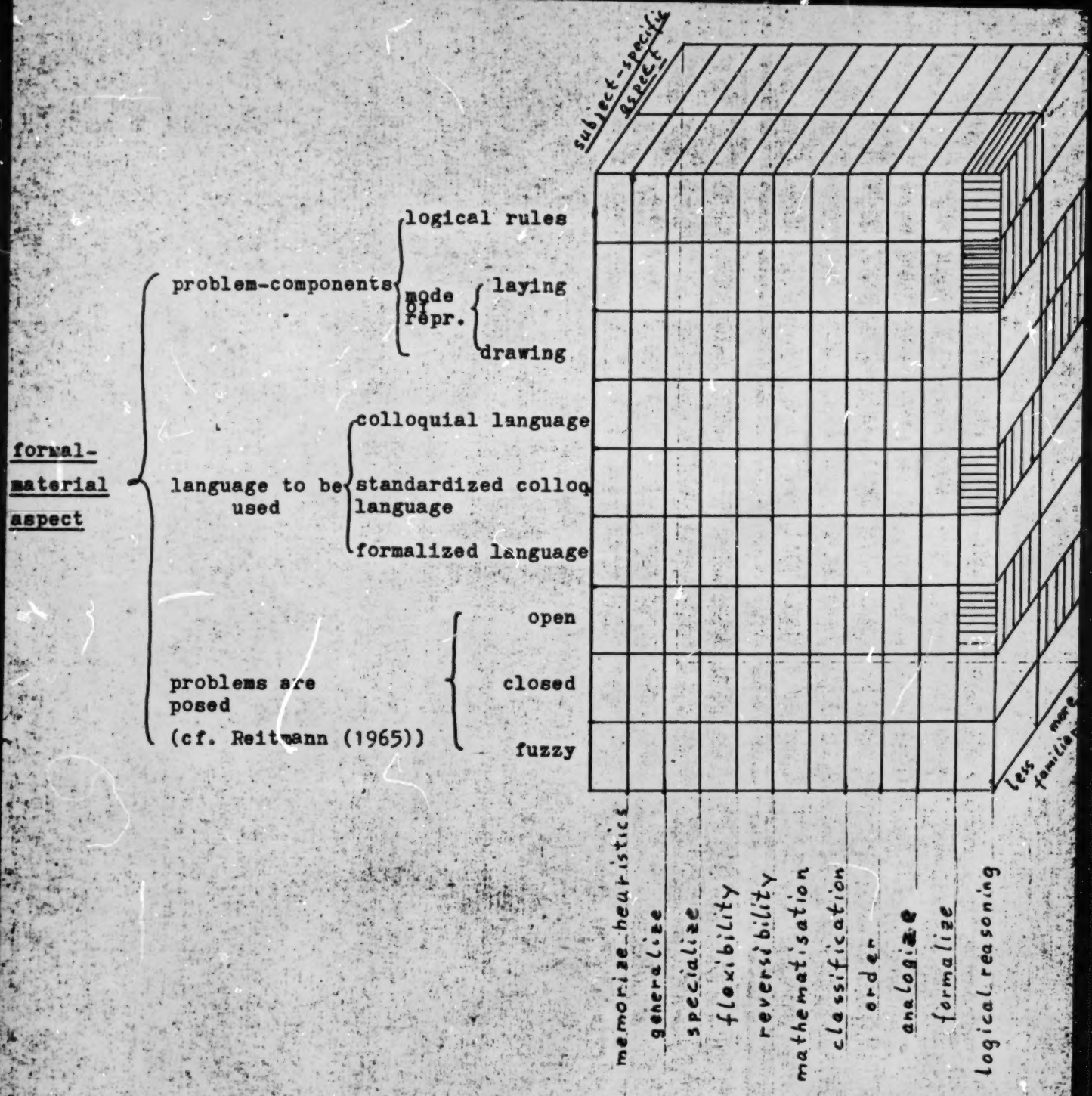
Fig. 3

Fig. 4



Integration of the problems from Aristotelian logic (Dörner/Lüer) into the classification system (cf. Fig 1)

Fig. 3



Integration of the problems from incidence geometry into the classification system (cf. Fig. 1, 3)

Fig. 4

The problem-score was composed in the following way:
For every problem 10 points were attainable at most.
They were given for

- realizing the problem (comprehending) (1 - 2 points),
- realizing useful axioms or theorems (1 - 3 points),
- correct application of these "operators" (1 point),
- realizing a useful proof-technique like proof by contradiction (1 - 2 points).

2.3 Procedure

The students had to solve the problems while thinking aloud within three single sessions. Each of these sessions lasted from one and a half to two hours and was videotaped. The study was conducted from November 1974 until June 1975.

It was possible for the subjects to write down their ideas, to make drawings or visualize by use of strings (representing the curves) and buttons (representing the points). The first way of visualisation was called "iconic", the second "enactive" representation. In contrast to the "classical" drawing visual situations can be changed continuously by the second possibility by moving a string or a button. These possibilities should be given because of goal 4 and 5 (cf. 1.5).

The subjects were assisted by a test-leader, whose only task was to keep the subject think aloud. His influence should be as trifling as possible and a list of interventions arranged in a flow-chard was standardized for this purpose. Otherwise there would have been the danger of analyzing the problem solving process of the test-leader instead of the subject.

The video-tape observations of the sessions from the three problems presented in 2.2 were translated into standardized protocols with the aid of a complex category system (cf. 3.1). The development of

this category system and the translation of the observations took more than ten months until April 1976.

The translated protocols were evaluated by use of correlation and sequence (Markov-chain) analysis.

The translation of the video-observations into standardized protocols was managed in the following way:

The events on the video-tapes were subdivided into small "observation-steps" which were defined to be the shortes units, to which could be given sense. The average duration of such step was some 8 seconds.

At most fifteen aspects were incorporated in every observation-step, which defined the "fundamental-activities" (cf. 1.4).

Such steps were separated, if a change occurred in at least one column.

This coding-procedure will be illustrated by an example in Table 1 and Table 2.

Table 1

Table 2

Table 1

Excerpt from a test protocol (subject female, age 21, Theorem 2, beginning)

1. Subject reads the text of Theorem 2 (transitivity of parallelism) very accurately and slowly (next steps literal translation from German)
2. "Mh - well, if k and l are some arbitrary curves k and l, l and s, then they have ...
3. - that would be a curve - if that (subject is drawing a curve) would be k
4. - if that (drawing a curve parallel to k and distinct from k) would be l
5. - then this would be s (drawing s like k) and then ... "

Table 2

Excerpt from a standardized protocol (translation of the protocol from table 1)

	verb.	writ.	int.	gest.	draw.	int.	lay.	int.	loc.	log.	glob.	aff.	mistake log.	mistake int.	sol./t
1.	WA		.									V			
2.	LIETPTV2								LSVAA2		LSVAA				
3.	LIETPTV21				K	LIETPTV21			LSVAAV2						
4.	LIETPTV21				KP	LIETPTV2			LSVAAV2	LVP2V2			LFP2	E6P	
5.	LIETPTV2				KP	LIETPTV21			LSVAAV21	LLYPPLVN2			WLFP2V)	WE6PC2	

There will now be given some explanation of Table 2. The numbers of the observation-steps in Table 1 correspond to those in Table 2. The verbal expressions of the subject were put into the first column labeled "verb." with interpretation as little as possible. E. g. "WA" in step 1. means: the subject reads the text of the problem. LIETPTV2... in step 2. - 5. means in this column: the subject repeats a part of the problem in her own words, namely the supposition of Theorem 2, part (i) or part (ii) (cf. 2.2). The abbreviations used here, are due to the category-system described in section 3.1 more explicitly.

If the subject was writing something (often at the same time when she was thinking aloud), a suitable abbreviation (like in the first column) was written into the second column (writ.). The supposed reason of this writing (interpretation) was written into the third column (int.).

The next column (gest.) was reserved for all gestures and nonverbal and nonvisualizing behavior. In that case the respective column is empty (Table 2). E. g. "WZ" would mean: the subject was looking at or pointing on a drawing to that moment.

The fifth column (draw.) was intended for all drawing-activities, the sixth for its interpretation (int.). E. g. in step 3. the subject drew a curve (K), by this representing the first part of the supposition of Theorem 2 (LIETPTV2I). In step 4. and 5. the subject drew parallel curves (KP) which should represent parts of the supposition of the problem (LIETPTV2I and LIETPTV2II respectively).

The same abbreviations would have been written into column 7 (lay.) resp. 8 (int.), if the subject would had represented the supposition by the laying of strings instead of drawings. Especially these columns were important with regard to goal 4 (cf. 1.4).

The content of the previous columns and of the 12th column (aff.) was concentrated in the 9th column (loc.). This column should contain the probable function of the present activities of the respective step in the way, the subject saw herself (local interpretation). E. g. LSVAAV2... (step 2.-5.) means: the subject tries to understand a part of the supposition of the problem (theorem 2).

Logical operations were registered into the 10th column (log.). The declaration of an action as a logical operation implied more interpretation of the evaluator, of course. E. g. LLV means: application of the modus ponens. So LLVDP2(AV2Ī) means: application of the second part of the defintion of parallelism to the first part of the supposition of the actual problem (theorem 2). Two curves were defined to be parallel, if they (i) were equal or (ii) did not have a point in common.

The content of several observation-steps should be summarized as one "essential" step in the 11th column (glob.). As the opinions of the two evaluators differed rather strongly in what could be condensed and defined as an "essential" step the content of this column was not incorporated into the evaluation process.

The 12th column, labeled aff., contained the affective behavior of the subject to that moment. E. g. V in step 1. means: the subject was hesitating. Particulary "aha-experiences" (subjective solution-ideas, see below objective solution-ideas) and perplexity was noted here.

The logical content of a mistake was fixed in column no. 13 (log.), its possible interpretation in column no. 14 (int.). E. g. the subject made an incomplete application of the definition of parallelism (so LFP2 in "log.") in step 4.,,which is probably due to some "prehistoric" experience about parallel lines which must be distinct (so EGP in "int."). As the subjects repeats this mistake from step 4. in step 5., there had been noted WLFP2(4) and WEGP(4) respectively into the corresponding columns.

The very important "objective" solution ideas and the time, which was needed up to now, was recorded into the last column (sol./t.). "Objective" means: in the sense of the evaluator. Example: the subject expresses the intention to make a case-distinction into parallel and identical and parallel and not identical curves (theorem 2).

These objective solution-ideas need not be the same like an "aha-experience", which is registered into the column "aff." (see above), because he might be on the wrong track.

So far the report about the rather complex process of building up a standardized protocol (Table 2). In this way 60 protocols were constructed including some 6500 observation-steps, where every step contained as many as 15 different fundamental activities described and abbreviated by a special category system (cf. 3.1). In this way more than one million alphanumerical signs were punched on more than 20,000 cards. These largescale protocols wherein nearly every information was preserved, enabled the analysis of the stated goals and even more treatment of further questions. This seems to be a useful approach for an exploratory study (cf. 1.3 f)).

The statistical evaluation was managed by an UNIVAC 1108-computer. Some more information about this process will be given now.

The fundamental activities - built by the concepts of the category system and noted into the 15 columns of the standardized protocols - were condensed into some 180 variables. These variables included the total number of observation-steps for all three problems, the operation time, the number of interruptions by the test-leader and the problem-score for every subject (cf. 2.2). Because the frequencies of the variables were depending on the number of observation-steps for every subject, partial correlation was to be used with respect to this variable. The entire correlation-matrix cannot be given here

because of the large number of variables.

Correlation analysis was used to find global relationships between variables and to determine connections which were possible to be investigated through sequence analysis. Sequence analysis was used to explore causal relationships.

The sequence analysis was managed in the following way:

a) All 60 protocols were joined together resulting in a large protocol, consisiting of some 6500 observation-steps.

b) As there were as many as 15 parallel sequences (cf. Table 2), these had to be condensed into one. The construction process of the category-system (cf. 3.1) showed the content of the first column (verb.) to be most reliable and informative. Depending on the research objectives (cf. 1.5) the content of the first column was exchanged e. g. for the content of the column "sol./t.", as far as a solution idea occurred in it. So local conditions for successful problem solving could be analyzed (cf. goal 3). According to this procedure the content of the column 2, 4, 6, 8, 10, 12, or 14 were inserted into the first column in that succession (priority), which was given by the respective objectives.

c) After this transformation of 15 parallel sequences into one, a futher shortening of this "super-sequence" took place:
Any sequence of <u>equal</u> consecutive fundamental activities (which were now the same like observation-steps) was identified by only <u>one</u> step of this kind. So the whole sequence was pushed together to a sequence consisting of <u>different</u> fundamental activities only.

d) Now the relative frequencies of all remaining fundamental activities in this super sequence were determined. By multiplying these frequencies of two fundamental activities, the expected probability for their consecutive occurence was estimated, providing random distribution. Subsequently the relative frequency of this respective

Category-System to describe "subjektive" and "objektive" aspects of problem-solving processes

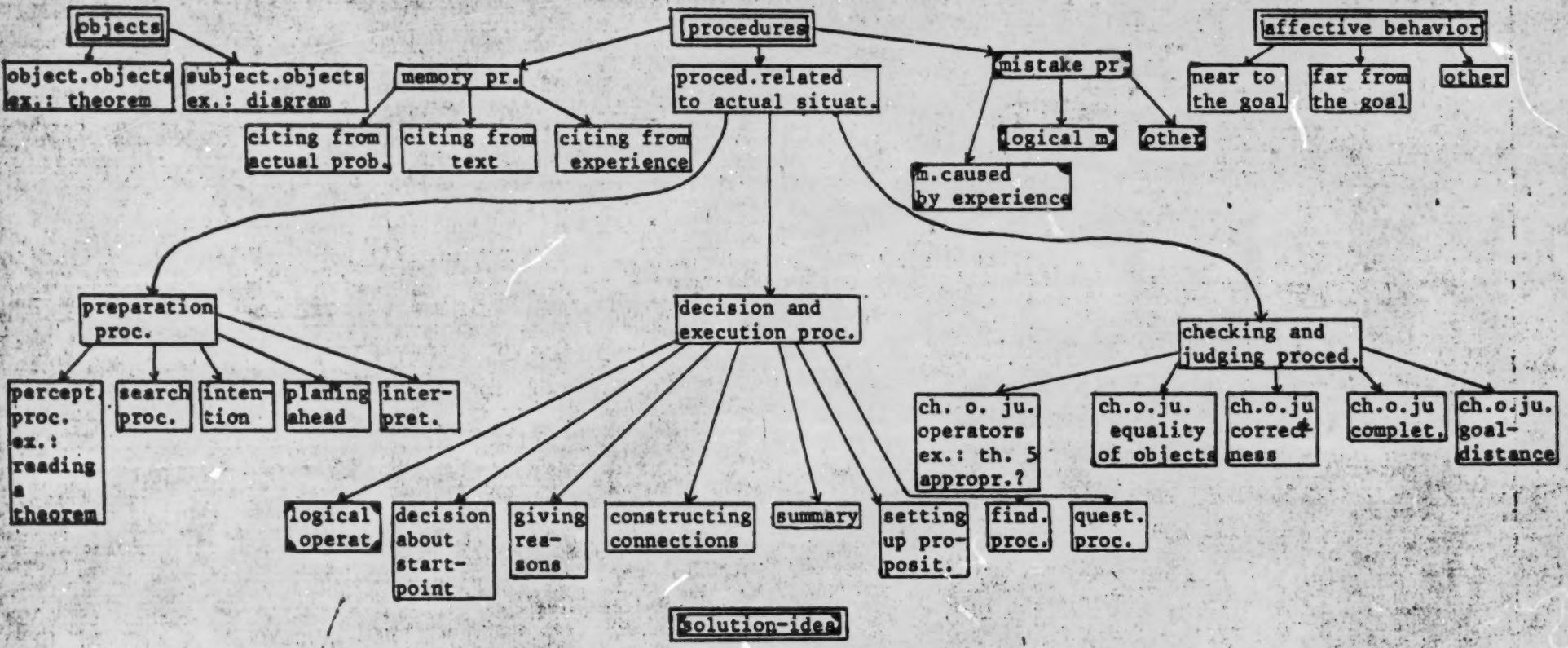


Fig. 5

two-member sequence was determined. It was decided by use of the binominal test, whether this sequence occurred more often than it could be expected by chance. As the bindminal test has to be applied up to 4300 times, the error probability accumulated. So the error probability for one test was chosen to $\alpha=0.00001$, so that a total of $1-(1-\alpha)^{4300} \approx 0.04$ for all tests was not exceeded.

An analogous procedure was applied to 3-, 4-, 5- and 6-member sequences.

So far the description of the research procedure.

3. Results

3.1 The category-system

During the development of the category-system care was taken to obtain objectivity by having a second evaluator for some protocols. For reasons of validity some of the standardized protocols were also presented to some of the subjects, after all single-sessions were brought to an end. These methods ensured, as far as possible, that the evaluator could not project his own way of thinking into his observations. In both cases a sufficient degree of agreement was obtained. The degree of correspondence lay between 77% and 94%. The result was a category-system resembling the third version of Kilpatrick's (1967) coding-scheme. A rough draft of this system is presented as a schematic diagram in Fig. 5.

Fig. 5

The diagram is constructed in a hierarchical way: The generic terms are at the top, boxes on similar level often contain concepts on a similar level. But in general even the lowest boxes contain only the label for a lot of more specialized concepts. The whole category-system contains some 340 concepts, which were composed to more than 4000 different fundamental activities in the evaluation-process.

Some explanation of this category-system has already been given in 2.3 (cf. Table 1, 2). Its main parts are "objects" and "procedures", which operate on these objects. A further example may demonstrate their joint operation:"Because of this (subject refers to his own drawing, consisting of a curve and a point near to it, which is called a "subjective" object and was constructed in step 10., therefore it is abbreviated by Z10) and the second part of the definition of parallelism (s. refers to part (ii) of the definition of parallelism, abbreviation P2, an "objective" object), I can draw this parallel curve through this point." This statement would be abbreviated by LH(Z10,P2) (subject constructs a connection between Z10 and P2, cf. the box "constructing connection" in Fig. 5), which can be inserted into the columr "verb." and "loc." (cf. Table 2). In "log." LLVDP2(z10) would be inserted, which is an interpretation of the evaluator, of course.

Some further statements of subjects are given to explain the remaining sub-concepts.

Citing from actual problem (recall or recapitulation): "I mentioned above, that ..." (GV...). Citing from text: "Curves were defined to be parallel, if ...." (GT...). Citing from experience: "A line is always determined by two points " (GE...). Search procedure: "Now I am looking for a suitable theorem." (LST...). Intention: "I want to apply Axiom 3 now." (LAA3). Planning ahead : "I want to prove the theorem by use of Axiom 3 and the definition of parallelism." (LV(A3,P2,S2)). Interpretation: "that means ..." (LI...).

Decision about starting point: "I want to start with axiom 3 now." (LESA3). Giving reasons: "That is because of axiom 1." (LBA1). Summary: " I have now reached the following:...." (LZ...). Setting up a proposition: "And now the following statement is true:..." (LT...).Finding procedure: "That's the theorem I was looking for !" (F...). Questioning procedure: " May I now apply axiom 2 ?" (FOA2). Checking equality of objects: "Is this the same like that ?"(PÜ...). Judging equality of objects:"Yes, that's the same." (UÜ...). Checking correctness: "Well, that's, what I have to check now!"(PK...). Checking completeness:"Haven't I any forgotten ?" (PV...). Judging goal-distance: "That is no progress !" (ZH).

Finally some examples concerning "affective behavior" (cf. "aff." in Table 2). Near to the goal: "That's it !" (aha-experience, abbreviated by "I"). Far from the goal: Subject is perplexed, helpless (R). Other: Subject is hesitating (V, cf. Table 2).

3.2 Search for a fundamental rhythm

After a detailed discussion between the author and Prof. Dörner all standardized protocols were translated into the categories of Dörner (1974) according to the procedure described in 2.3, a) - d). So the result was a "super-sequence", consisting of 4066 observation-steps, formed by 18 different categories. Sequence-analysis was carried out to determine those 2-, 3-, and 4-member sequences which occured more often than it could be expected by chance. The result of this analysis is presented in Table 3, 4, 5, 6 and Figure 6.

Table 3

Table 4

Table 5

Table 6

Fig. 6

Table 3

Explanation of categories used by Dörner (1974), their absolute (f_a) and relative (f_r) frequencies

Abbrev.	Explanation	f_a	f_r
LKOPEX	external (by writing or visualizing) application of an operator (theorem, axiom or definition)	141	0.0347
LKOPIN	internal application of an operator (verbalizing only)	324	0.0797
LKROP	reverse application of an operator (logical replication)	20	0.0049
LKZWI	construction of an intermediate goal	32	0.0079
LPCIRC	vicious circle	2	0.0005
LPKOM	checking applicability of an operator	541	0.1331
LPNAE	checking goal-distance	444	0.1092
LVSOPEX	external search of an operator	724	0.1781
LVSOPIN	internal search of an operator	50	0.0123
LVSTA	choice of a starting point	343	0.0843
LVWA	reading the problem	76	0.0187
(the following categories were not used by Dörner)			
GV	subject repeats own statement	50	0.0123
LA	subject expresses intention	107	0.0263
LB	giving reasons	121	0.0297
LII	subject interprets own statement	241	0.0593
LSS	searching a starting point	358	0.0880
X	request of the test-leader to think aloud (This category was omitted in the following evaluation-process, as was done by Lüer and Dörner, too)	437	0.1075

Table 4

2-member sequonces which occured more often than it could be expected by chance ($\alpha = 0.0000\%$, rel. frequ. $\geq 0.4\ \%$)

sequence	f_a	f_r in %
LKOPEX → LII	31	0.77
LKOPIN → LPNAE	74	1.85
LKOPIN → LB	33	0.82
LKOPIN → LII	55	1.37
LPKOM → LVSOPEX	205	5.11
LPNAE → LSS	73	1.82
LVAB → LSS	17	0.42
LVSOPEX → LKOPIN	97	2.24
LVSOPEX → LPKOM	262	6.54
LVSTA → LII	42	1.04
LVWA → LVSTA	29	0.72
LA → LVSTA	29	0.72
LSS → LVSTA	57	1.42
		$\Sigma = 23.65\ \%$

Table 5

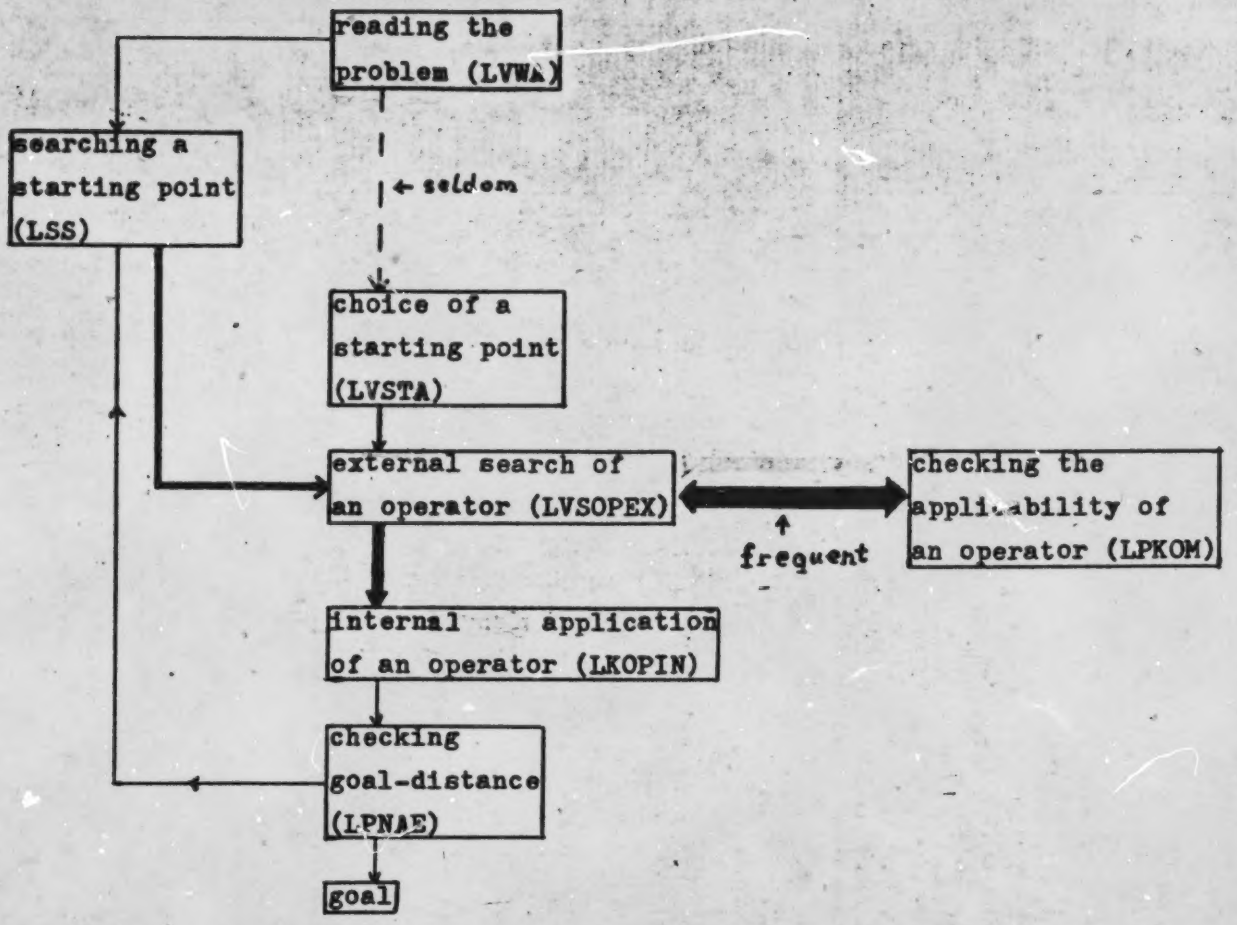
3-member sequences which occurred more often than it could be expected by chance ($\alpha \leq 0.00001$, rel. frequ. ≥ 0.15 %)

sequence						f_a	f_r in %
LKOPEX	-	LII	-	LKOPEX		6	0.15
LKOPIN	-	LII	-	LKOPIN		13	0.33
LKZWI	-	LVSOPEX	-	LPKOM		7	0.18
LPKOM	-	LKOPIN	-	LPNAE		17	0.43
LPKOM	-	LVSOPEX	-	LKOPIN		30	0.76
LPKOM	-	LVSOPEX	-	LPKOM		84	2.13
LVAB	-	LVSOPEX	-	LPKOM		8	0.20
LVSOPEX	-	LKOPEX	-	LII		11	0.28
LVSOPEX	-	LKOPIN	-	LPNAE		24	0.61
LVSOPEX	-	LKOPIN	-	LB		12	0.30
LVSOPEX	-	LKOPIN	-	LII		17	0.43
LVSOPEX	-	LPKOM	-	LPNAE		30	0.76
LVSOPEX	-	LPKOM	-	LVSOPEX		126	3.19
LVSTA	-	LVSOPEX	-	LPKOM		24	0.61
LVWA	-	LVSTA	-	LSS		6	0.15
LA	-	LKOPIN	-	LII		7	0.18
LA	-	LVSTA	-	LVSOPEX		11	0.28
LII	-	LKOPEX	-	LII		6	0.15
LII	-	LVSOPEX	-	LPKOM		221	0.53
LSS	-	LVSOPEX	-	LPKOM		32	0.81
LSS	-	LVSTA	-	LVSOPIN		6	0.15
LSS	-	LA	-	LVSTA		7	0.18
							$\Sigma = 12.79\%$

Table 6

4-member sequences which occurred more often than it could be expected by chance (α =0.00001, rel. frequ.≥ 0.1 %)

sequence								f_a	f_r in %
LKOPIN	-	LII	-	LVSOPEX	-	LPKOM		5	0.13
LKZWI	-	LVSOPEX	-	LPKOM	-	LVSOPEX		4	0.10
LPKOM	-	LVSOPEX	-	LKOPEX	-	LII		4	0.10
LPKOM	-	LVSOPEX	-	LKOPIN	-	LPNAE		10	0.26
LPKOM	-	LVSOPEX	-	LKOPIN	-	LII		5	0.13
LPKOM	-	LVSOPEX	-	LPKOM	-	LVSOPEX		50	1.29
LVAB	-	LVSOPEX	-	LPKOM	-	LVSOPEX		4	0.10
LVAB	-	LSS	-	LVSOPEX	-	LPKOM		4	0.10
LVSOPEX	-	LKOPEX	-	LII	-	LVSOPEX		5	0.13
LVSOPEX	-	LKOPIN	-	LII	-	LKOPIN		6	0.15
LVSOPEX	-	LKZWI	-	LVSOPEX	-	LPKOM		4	0.10
LVSOPEX	-	LPKOM	-	LVSOPEX	-	LKOPIN		220	0.51
LVSOPEX	-	LPKOM	-	LVSOPEX	-	LPKOM		48	1.23
LVSOPEX	-	LPKOM	-	LVSOPEX	-	LVSTA		13	0.33
LVSTA	-	LVSOPEX	-	LPKOM	-	LVSOPEX		12	0.31
LVSTA	-	LII	-	LVSOPEX	-	LPKOM		5	0.13
LVSTA	-	LSS	-	LA	-	LVSTA		4	0.10
LVWA	-	LSS	-	LVSOPEX	-	LPKOM		4	0.10
LB	-	LPKOM	-	LVSOPEX	-	LPKOM		5	0.13
LII	-	LVSOPEX	-	LPKOM	-	LVSOPEX		8	0.21
LSS	-	LPNAE	-	LSS	-	LVSOPEX		7	0.18
LSS	-	LVSOPEX	-	LPKOM	-	LVSOPEX		11	0.28
LSS	-	LA	-	LVSTA	-	LVSOPEX		4	0.10
									Σ=6.10


reading the problem (LVWA)
searching a starting point (LSS)
seldom
choice of a starting point (LVSTA)
external search of an operator (LVSOPEX)
checking the applicability of an operator (LPKOM)
frequent
internal application of an operator (LKOPIN)
checking goal-distance (LPNAE)
goal

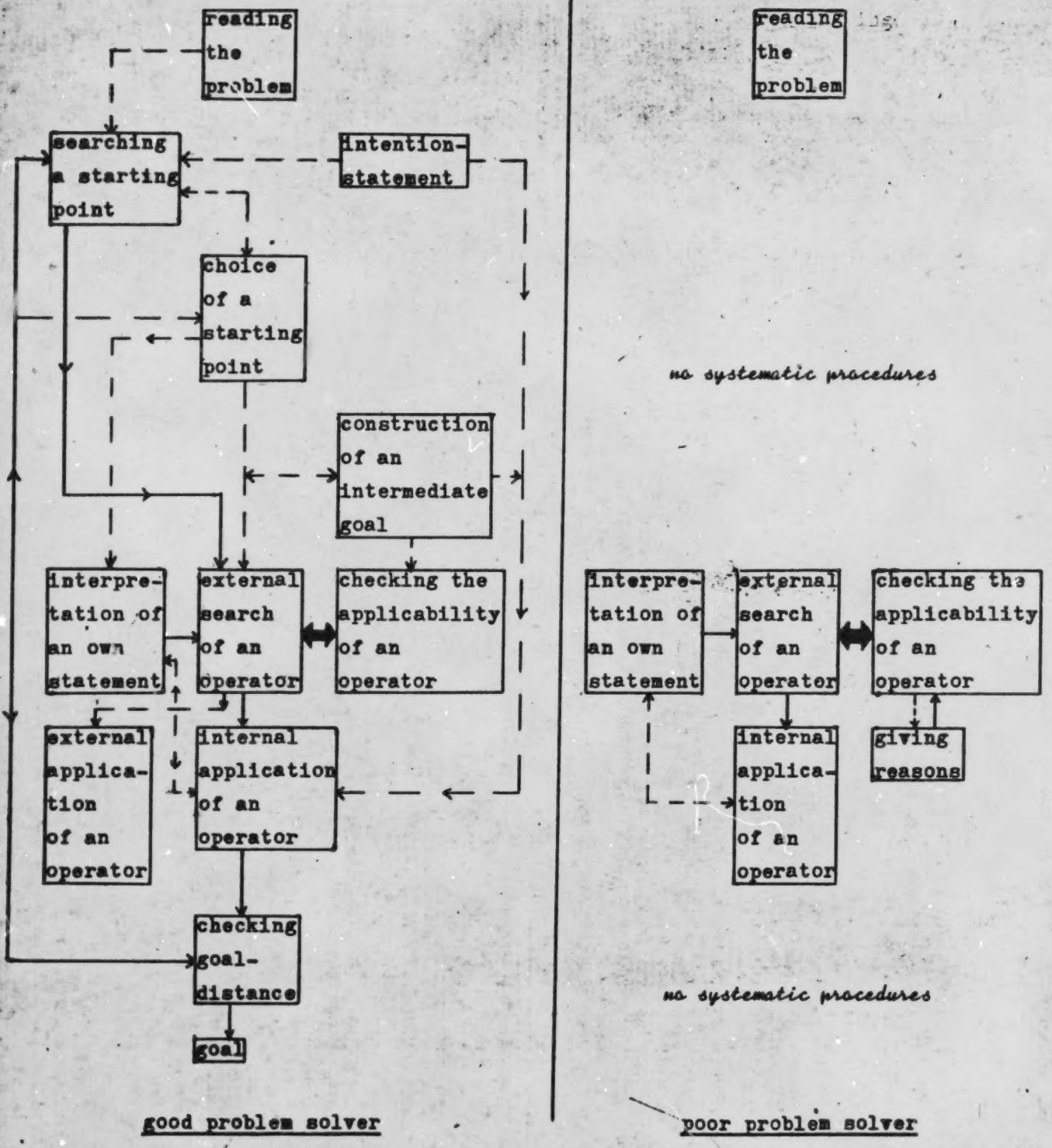
Fundamental rhythm involved with problems from incidence geometry (cf. Fig. 2, construction by superposition of the sequences from table 4 - 6, the thickness of the arrows represents the frequency of the transition approximately)

Fig. 6

As it was stated in 1.5 objective no. 3, a possible connection between the pattern of the fundamental rhythm and the problem solving success should be searched, too. This was managed in the following way:

The whole group of 20 students was subdivided into groups of equal size consisting of 10 members respectively. The first group was composed of all subjects with a high problem score (≧ 11 scores), the second group was formed by all subjects with a low problem score (≦ 10). To each of these groups the same procedure was applied like to the whole group (cf. Table 3 - 6, Fig. 6). Every group was formed by 5 female and 5 male students. The result of this analysis is presented in Fig. 7.

Fig. 7



Comparison of the fundamental rhythm of good and poor problem solvers

(cf. Fig. 6)

Fig. 7

3.3 Global and local conditions for successful and non-successful problem solving

At first global connections between the problem score and other variables are presented by (partial) correlation analysis.
The entire intercorrelation table cannot be given here because of reasons of space (78 pages).

At first possible conditions for the problem solving success are presented:

Correlation between problem score and	r=
1. goaloriented intentions (e. g. subject intends to prove an intermediate proposition oriented to the goal)	0.52
2. planning ahead	0.45
3. intention to make a new beginning	0.49
4. intention to apply a special proof-strategy (e. g. proof by contradiction)	0.43
5. conscious intentions (summary of the variables 1. - 4.)	0.72
6. intentions (summary of the variables 1. - 5. and "intention, to apply or to search for an operator")	0.56
7. modus ponens (highest correlation among all logical operations)	0.38
8. reference to the test material	0.48

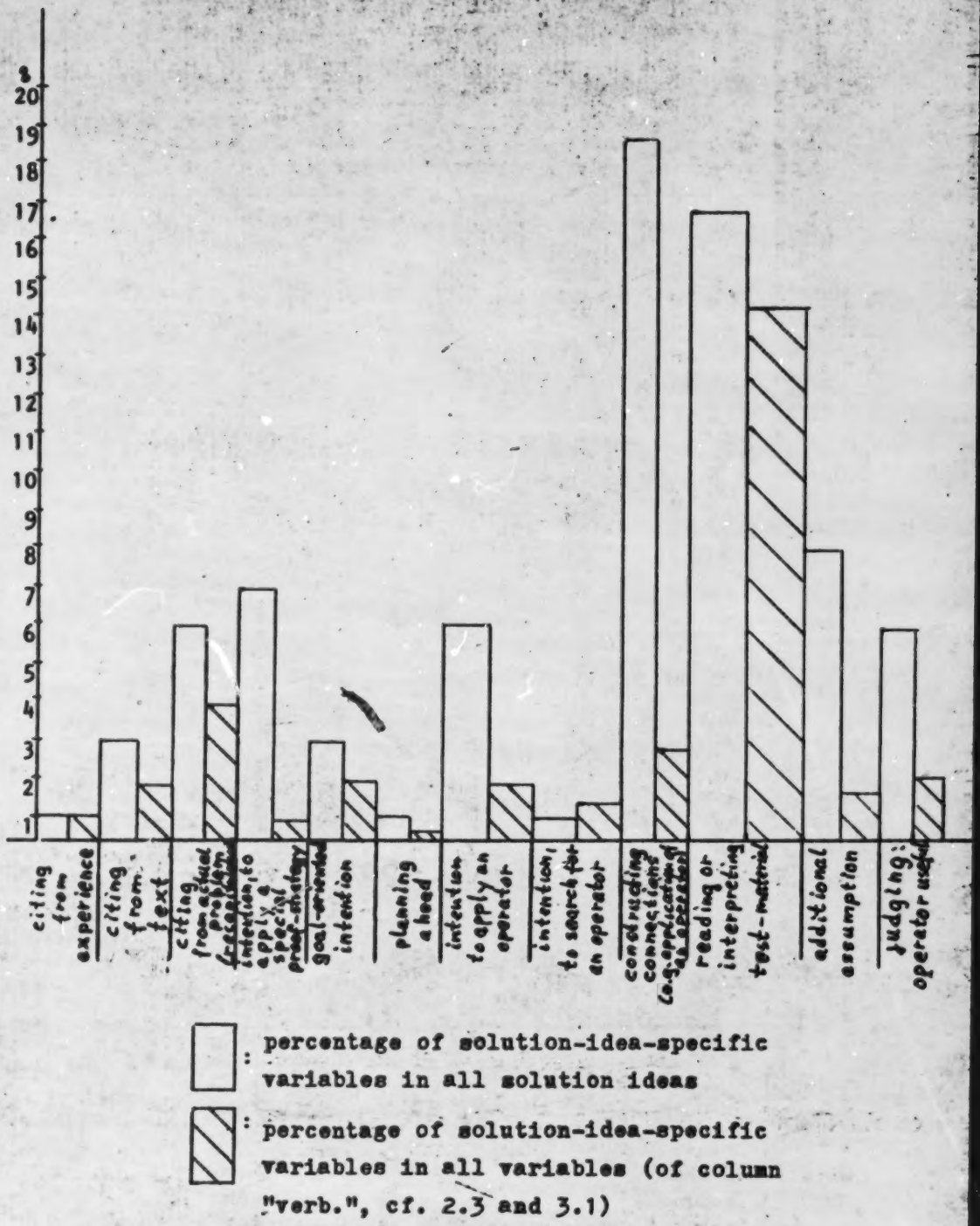
High positive intercorrelation (up to 0.7) could be registered nearly between all of these (the independent only, of cours)variables.

There was no variable which had a negative correlation less than -0.38 (reference to the proposition of the actual theorem) with the problem score (except mistakes, of course).

Recapitulations of own statements (GV) showed no significant correlation with the problem score (r=0.06). But there was a substantial correlation between recapitulations and interventions of the test-leader (r=-0.67), respectively short pauses (10 seconds an average) of the subject (r=-0.64).

Now <u>local</u> connections between objective solution ideas and other variables are presented. 102 solution ideas could be observed registered in the column sol./t. (cf. Table 2). At first the frequencies of those variables (fundamental activities in the column "verb.") were determined by which the respective solution idea was expressed. These frequencies were set into relation to all solution ideas and to all fundamental activities occurring in the column "verb.". So it could be checked by comparing these relative frequencies whether a variable appeared more often or more seldom in connection with solution ideas than with other fundamental activities (Fig. 8).

Fig. 8



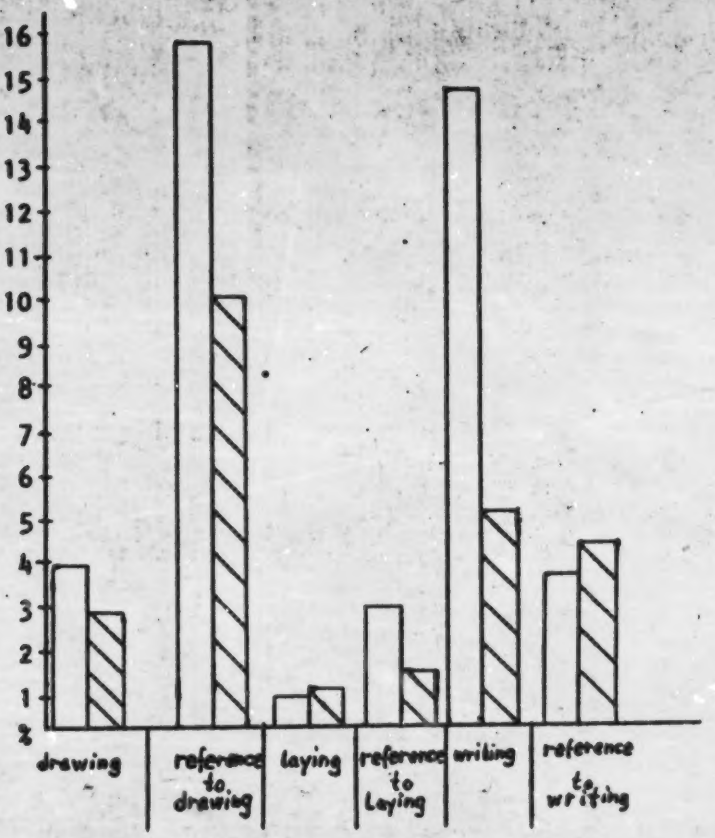
Frequency distribution of variables involved with solution ideas

Fig. 8

Fig. 8 shows that nearly all variables occur disproportionately often with solution-ideas. Some of these variables already correlated positively with the problem-score. More interpretation will be given in the discussion section (4).

In an analogous way the connections between the different modes of representation and solution-ideas were analyzed. The result is presented in Fig. 9. The percentage of perception-specific variables in "all" variables was determined after inserting these variables into the column "verb." (Table 2) whenever they occured and following computing the frequencies of the variables in this column.

Fig. 9



: percentage of perception-specific variables in all solution-ideas

: percentage of perception-specific variables in all variables (of column "verb."; cf. 2.3, 3.1)

Frequency distribution of perception-specific variables involved with solution-ideas

Fig. 9

Fig. 9 shows, that reference to drawings and "layings" and especially writing occur disproportionately often with solution-ideas.

Sequence-analysis should determine those variables which immediatly occured before solution-ideas more often, than it could be expected by chance. These variables might be interpreted as possible reasons for solution-ideas and so for the problem solving success.

Table 6

Table 7

Table 6

Significant 3-member sequences including solution-ideas

(α=0.00001, abs. frequ. $f_a \geq 2$)

sequence	f_a	f_r in %
DRAW - REFDRAW - IDEA	4	0.08
REFDRAW - IDEA - REFDRAW	7	0.14
REFWRIT - IDEA - REFWRIT	4	0.08
RECAPIT - REFWRIT - IDEA	3	0.06
REFTEXT - IDEA - NEARGOAL	2	0.04

Table 7

Significant 4-member sequences including solution-ideas

(α=0.00001, abs. frequ. $f_a \geq 2$)

sequence	f_a	f_r in %
DRAW - REFDRAW - IDEA - DRAW	2	0.04
REFWRIT - WRIT - IDEA - REFWRIT	2	0.04
WRIT - REFWRIT - WRIT - IDEA	3	0.06
WRIT - REFPROB - IDEA - WRIT	2	0.04

Tables 6 and 7 show that operations concerning drawing and writing are not only directly connected with solution-ideas (cf. Fig. 9), but also immediately precede solution-ideas more often than it could be expected by chance. Another priority in the identification process (more stress on the content of the column "verb.") before applying sequence-analysis turned out that recapitulation, reference to the test material (REFTEXT) and to the actual problem (REFPROB) precede solution-ideas significantly often.

Sequence-analysis showed no variable which immediately preceded impasses more often than it could be expected by chance.

3.4 The part of different modes of representation in the problem solving process and their influence on the problem solving success

Local analysis, esp. sequence analysis already proved drawing and writing (resp. reference to these operations) to be significant for solution-ideas and therefore for the problem solving success (cf. 3.3).

In contrast to this there were no correlations greater than 0.33 (absolute value) between different modes of representation and problem-score:

Correlation between problem-score and	r=
writing	0.04
reference to writing	0.08
drawing	0.02
reference to drawing	0.16
laying	-0.15
reference to laying	-0.33

Significant sequences which included a visualizing operation were picked out to determine possible reasons and consequences for changes in the mode of representation. Sequence-analysis was carried out with respect to different priorities (when pushing together the 15 parallel sequences into one, cf. Table 2). One result is presented in Table 8.

Table 8

Table 8

Significant 3-member sequences to determine possible "prehistory" and "posthistory" of visualizing operations (α=0.00001, abs. frequ. $f_a \geq 2$)

sequence				f_a	f_r in %
DRAWING	-	REFER.WRITING	- DRAWING	4	0.10
DRAWING	-	READ.ORINT.PROB	- REFER.DRAWING	8	0.20
DRAWING	-	READ.ORINT.TEXT	- REFER.DRAWING	9	0.23
REFER.DRAWING	-	READ.ORINT.PROB	- REFER.DRAWING	15	0.38
REFER.DRAWING	-	READ.ORINT.TEXT	- REFER.DRAWING	30	0.76
READ.ORINT.PROB	-	DRAWING	- READ.ORINT.PROB	11	0.28
READ.ORINT.PROB	-	INTENT.OR.DRAW	- READ.ORINT.PROB	7	0.18
INTENT.SEARCH.OP	-	REFER.DRAWING	- READ.ORINT.TEXT	6	0.15
READ.ORINT.TEXT	-	REFER.DRAWING	- USELESS	7	0.18
READ.ORINT.TEXT	-	REFER.DRAWING	- READ.ORINT.TEXT	32	0.81
IMPASS	-	READ.ORINT.PROB	- DRAWING	2	0.05
IMPASS	-	READ.ORINT.TEXT	- DRAWING	11	0.28
INTENT.APPLIC.OP	-	REFER.LAYING	- LAYING	2	0.05
WRITING	-	GOALORIE.INTENT	- REFER.WRITING	3	0.08
WRITING	-	SUMMARIZE	- READ.OR.INT.PROB	2	0.05
WRITING	-	READ.OR.INTPROB	- CHECK.GOAL.DIST	4	0.10
WRITING	-	READ.OR.INTTEXT	- WRITING	12	0.30

3.5 Determination of problem solving styles

Already by a rough analysis of the data the sample of this study could be separated into two nearly disjoint groups.:
The first group was determined by those subjects, who (nearly) exclusively visualized by means of drawings. This group was formed by 12 subjects and was named iconic group.

The second group was determined by those subjects, who (nearly) exclusively visualized by means of strings and buttons. This group was formed by 7 subjects and was called enactive group.

Only one person neither drew figures nor laid strings or buttons. Further differences and characteristics of these problem solving styles were determined by use of (partial) correlation. As there could be stated r=-0.70 between "drawing or refering to drawing" and "laying or refering to laying", the differences between these activities are underlined. Correlations are presented between enactive resp. iconic activities and other variables in Table 9.

Table 9

Table 9

Partial correlation between iconic resp. enactive activities and other variables

		iconic style			enactive style			cognitive style (reflexive/impulsive)		
		drawing	refer. to drawing	drawing or refer. to drawing	laying	refer. to laying	laying or refer. to laying	time	pause	mistake incon.
cognit. style	1. time	0.15	0.15	0.15	-0.35	-0.36	-0.37	-	0.39	-0.44
	2. pause	0.35	0.31	0.32	-0.25	-0.29	-0.28	0.39	-	-0.22
	3. mistakes (general)	-0.10	-0.20	-0.18	0.55	0.53	0.55	-0.40	-0.20	-
	4. esp. mistake.incor.	-0.34	-0.45	-0.44	0.64	0.61	0.63	-0.44	-0.22	-
	5. esp. mist.colloq.lang	-0.50	-0.36	-0.39	0.54	0.55	0.55	-0.09	-0.23	-
	6. esp. m. by percept. (functional fixtness)	-0.30	-0.35	-0.34	0.82	0.68	0.75	-0.09	-0.16	-
	7. esp. confusion	-0.35	-0.30	-0.31	0.56	0.50	0.53	-0.55	-0.36	-
	8. esp. incompl. reason	0.40	0.42	0.42	-0.02	0.02	0.01	-0.46	0.03	-
	9. esp. m. from extern. experience	0.40	0.42	0.42	-0.24	-0.21	-0.23	0.10	0.31	-
	10. esp. wrong reverse inference	0.26	0.31	0.30	0.17	-0.04	0.04	-0.12	0.07	-
	11. correct. of mistake	-0.33*	-	-	0.35*	-	-	-0.31*	-0.31*	-
	12. esp. correct.confus.	-0.69*	-	-	0.85*	-	-	-0.37*	-0.46*	-
	13. esp. correct.mistak. conc. colloq. lang.	-0.37*	-	-	0.16*	-	-	-0.36*	-0.11*	-
verbal factor	14. recapitulation	-0.33	-0.31	-0.32	0.45	0.59	0.55	-0.64	-0.64	0.56
	15. interpretation of own statement	-0.24	-0.31	-0.30	0.47	0.60	0.56	-0.40	-0.27	0.52
	16. intervention of test-leader	0.25	0.17	0.19	-0.28	-0.31	-0.31	0.56	0.79	-0.27
	17. writing	-0.13	-0.24	-0.22	0.09	-0.11	-0.03	0.24	0.16	0.31
	18. ref. to writing	-0.30+	-0.47+	-	0.43+	0.35+	-	-0.35+	-0.42+	0.59+
intervention	19. planning ahead	-0.49	-0.50	-0.51	0.26	0.27	0.27	-0.43	-0.45	0.23
	20. intention: proof	-0.31	-0.20	-0.23	-0.10	0.01	-0.03	-0.23	-0.40	0.02
	21. intention: goal-ori.	-0.16	-0.03	-0.06	0.25	0.17	0.20	0.03	-0.37	-0.01
	22. intention: operappl.	-0.50	-0.37	-0.41	-0.01	0.26	0.16	-0.07	-0.48	-0.12
	23. intention: understa.	0.51	0.57	0.57	-0.31	-0.34	-0.34	-0.19	0.29	-0.01
judging or checking	24. judg.: opera. useful	-0.45	-0.38	-0.40	0.65	0.69	0.69	-0.44	-0.40	0.45
	25. judg.: reasoning incomplete	-0.49	-0.58	-0.57	0.71	0.77	0.77	-0.51	-0.19	0.69
	26. judg.: reasoning complete	-0.41	-0.41	-0.42	0.39	0.58	0.52	-0.12	-0.43	0.13
	27. checking goal-dista.	-0.22	-0.11	-0.14	0.41	0.25	0.32	0.15	-0.11	0.10
	28. logical operation (general)	0.04	-0.10	-0.07	0.21	0.27	0.26	-0.25	-0.42	0.32

Table 9 (continuation)

		iconic style			enactive style			cognitive style (reflexive/impulsive)		
		drawing	refer. to drawing	drawing or refer. to drawing	laying	refer. to laying	laying or refer. to laying	time	pause	mistake incorrect
logical operat.	29. esp. modus ponens	0.07	0.12	0.11	0.09	0.07	0.08	0.02	-0.27	-0.04
	30. esp. additional assumption	-0.02	-0.16	-0.13	0.16	0.34	0.28	-0.20	-0.28	0.30
	31. esp. negation	-0.03	-0.07	-0.06	0.52	0.42	0.47	0.07	-0.22	0.32
	32. esp. case-distinc.	-0.11	-0.24	-0.21	0.13	0.01	0.06	-0.29	0.13	0.41
orientat. to text-material	33. interpret. text (theorem or axiom)	-0.54	-0.47	-0.49	0.23	0.42	0.36	-0.34	-0.32	0.29
	34. recall text	0.42	0.36	0.38	-0.12	-0.10	-0.11	-0.29	0.39	0.32
	35. reading text	0.06	0.04	0.05	-0.08	-0.23	-0.18	-0.12	-0.25	-0.22
	36. interpret. or read actual problem	0.29	0.18	0.21	-0.17	-0.21	-0.20	-0.16	0.10	-0.17
	37. esp. reading prob	0.19	0.17	0.18	-0.43	-0.42	-0.43	-0.02	-0.07	-0.17
	38. age of the subject	0.30	0.33	0.33	-0.19	-0.16	-0.17	-0.37	-0.16	0.29
affecting behavior	39. feeling near goal	-0.41	-0.30	-0.33	0.30	0.12	0.20	-0.37	-0.55	0.10
	40. esp. sudden insig.	-0.50	-0.48	-0.50	0.70	0.54	0.61	-0.51	-0.33	0.50
	41. esp. subjective solution-idea	-0.33	-0.15	-0.20	0.22	0.09	0.14	-0.05	-0.29	-0.12
	42. esp. reading expressively	0.53	0.35	0.39	-0.48	-0.50	-0.51	-0.51	-0.29	-0.15
	43. feeling far goal	0.02	-0.02	-0.01	-0.04	-0.15	-0.11	-0.29	-0.06	-0.10
	44. esp. uncertain	0.13	0.06	0.08	0.31	0.21	0.25	-0.12	-0.05	0.28
	45. problem score	0.02	0.16	0.13	-0.15	-0.33	-0.27	-0.06	-0.17	-0.44

Remark: All correlations are partial with respect to the number of "observation-steps".

*: Partial correlation was computed with respect to the specific mistake additionally.

+: Partial correlation was computed with respect to "writing" additionally.

Some parts of the error-analysis were already anticipated in this way. A first rough analysis of the data suggests the assumption to the evaluator that the iconic style corresponds with the impulsive style and the enactive corresponds with the reflective style.

The impulsive style is defined by a high psychic tempo combined with a tendency to make a lot of mistakes in problem solving.

Reflective persons are slower in this respect and make fewer mistakes (cf. e. g. Rebhun (1973) and Walek (1972)).

The psychic tempo of the subject in this study was determined by the number of pauses ("pause" in Table 9, average-lenth of 10 seconds) and by the time ("time") required to solve all three problems (relative to all observation-steps). The most serious mistakes came from incorrect operations ("mistake incor.", excluding incomplete arguments).

The supposed connection between iconic and reflexive resp. enactive and impulsive style is supported by the correlation coefficients of the first three variables in line 1. - 3. (Table 9). For this reason intercorrelations had been computed between the variables defining the cognitive style (time, pause, mistake incor.) and all other variables respectively. Table 9 shows that nearly all variables (with some exceptions only) which correlate positively (negatively) with iconic variables (drawing, refer. to drawing, drawing or refer. to drawing) correlate positively (negatively) with the variables time and pause and negatively (positively) with mistake incor. . The analogous statement holds for the enactive variables.

3.6 Error analysis

Some information about mistakes was already given in the previous section.

There was a total number of 319 mistakes. At first their character and frequency is outlined.

A distinction was made between logical and material-psychological mistakes. The seccnd type rather often was nothing but an other aspect (interpretation) of the first one.

Confusions (e. g. confusion of supposition and proposition of a theorem, 7 1/2 % of all logical mistakes) and incomplete arguments for the existence or properties of (geometrical) objects (18 %) occurred rather often among the logical mistakes.

The most serious logical mistake was produced by the fact that many students did not recognize the existence-problem in Theorem 1 and Theorem 3 (7 1/2 %). Furthermore most other logical problems concerned the differences between supposition and proposition of a theorem and the security of the existence of objects described in a conditional clause.

Mistakes caused by geometrical experience (e. g. "parallel lines have to be distinct"), by perception (objects, which are drawn must exist!) and by the fuzzyness of the colloquial language occurred very often among the material-psychological mistakes.

Most mistakes correlated negatively with the problem-score, as one could have expected. But incomplete arguments (r=0.37) and par-force-solutions (r=/.48) showed positive correlation with the problem-score.

There was a positive correlation between all activities not directly connected with the test-material (e. g. changes in the mode of representation, interpretation of own statements, recapitualations) and all mistakes (r=0.64). Direct reference to the test material (e. g. reading the axioms or the actual problem) correlated negatively with all mistakes (r=-0.50).

Sequence-analysis did not show hardly any systematic connections between mistakes and other variables except changes in the mode of representation (Table 10).

Table 10

Table 10

Significant 3-member sequences to determine possible "prehistory" and "posthistory" of mistakes (α=0.00001, abs. frequ. $f_a \geq 2$)

sequence				f_a	f_r in %
DRAWING	-	INCOMPL.ARGUMENTS	- DRAWING	2	0.05
DRAWING	-	REPETITION MISTAK.	- REFER.DRAWING	4	0.09
SOL!IDEE	-	INCOMPL.ARGUMENTS	- CHECK.GOAL.DIST.	2	0.05
INTENT.OR.WRIT.	-	MISTAKE INCORRECT.	- REFER.WRITING	2	0.05
INTENT.OR.DRAW.	-	WRITING	- INCOPL.ARGUMENTS	2	0.05
INTENT.GOAL-ORIE.	-	MISTAKE INCORRECT.	- CORRECTION	2	0.05
WRITING	-	MISTAKE INCORRECT.	- REFER.WRITING	6	0.14
WRITING	-	MISTAKE INCORRECT.	- WRITING	4	0.09

Some additional information about the correction of mistakes is presented now.

Mistakes by confusion (26 % of all mistakes by confusion), incorporation of the proposition into the supposition (15 %) and mistakes caused by the colloquial language (10 %) were most often corrected by all subjects.

Except the problem-score (r=0.49) there were especially intentions and reference to the theorems or axioms correlating positively with the correction of mistakes (r=0.66 resp. r=0.54). Particularly there was a positive correlation between correction of mistakes and intention to search for appropriate theorems or axioms and to change the mode of representation (r=0.77 and r=0.69). "Feeling near to the goal" was connected with the correction of mistakes, too (r=0.55, esp. subjective solution-idea : r=0.63).

Negative correlations were stated to "interpretation of any object" (e. g. drawing, own statement, axiom, r=-0.67) and reference to (looking at without reading aloud) the text of the actual problem (r=-0.57).

Sequence-analysis only showed week connections between correction of mistakes and other variables. Only intentions and reference to the test-material preceded corrections somewhat more often than it could be expected by chance. So not only a global (cf. correlations) but also a local connection between these variables can be assumed.

4. Discussion

4.1 The fundamental rhythm

As one can see by comparison of Fig. 2 and Fig. 6, the pattern derived from the attempts of the subjects of this study to solve problems from incidence geometry looks like a "restricted" fundamental rhythm . Subjects seem to solve these problems in a less systematic way than those of Lüer (1973) and Dörner (1974). These differences might be explained in the following way:

(i) The influence of the test-leader was reduced to a minimum in the present study. His only function was to keep the subject think-aloud. In Lüer's and Dörner's study the test-leader had to correct the subject whenever he was making a mistake.

(ii) The problems of this study might be somewhat more difficult than those of Lüer and Dörner because of the logical rules and their application were not given to the subjects explicitly. Furthermore these rules were to be applied to some geometrical statements, which could interfere with former geometrical intuition. So the incidence of more trial-and-error (LVSOPEX - LPKOM) seems to be explainable.

(iii) The external structure of the logical problems (sequence of signs consisting of names of variables and logical signs like $\wedge, \vee, \rightarrow, \neg$) makes it more probable to list up intentions like "change $\vee$ into $\wedge$" or "eliminate brackets".

The analysis of Fig. 7 indicates that more successful problem solvers seemed to work on the problems more systematically than less successful subjects. It is possible, therefore, that Lüer's (Dörner's) fundamental rhythm might be represented by the more capable problem solvers.

4.2 Conditions for successful and non-successful problem solving

The correlations presented in 3.3 might be interpreted in that way that conscious actions oriented to intentions, to the goal and to the written material (theorems and axioms) are useful to solve problems. This supposition is confirmed by Fig. 8, too.

A high correlation between all summarized logical operations and all conscious intentions (r=0.57) supports a presumption of Hadamard (1949) that logical operations are connected with high consciousness.

The negative correlation between problem-score and reading the problem might be explained by the experience that the subjects tended to read the text of the problem again and again when they had no more ideas.

The stated correlations between recapitulations and pauses, resp. interventions by the test-leader, seems to be a confirmation of a hypothesis of Kilpatrick (personal communication) that "recapitulation" is a measure for the subject's ability (or willingness!) to think aloud. There was no significant correlation between these variables and the problem-score.

As pauses and interventions did not correlate with the problem-score as well, the ability to think-aloud (a kind of verbal factor) seems to be independent from problem solving success.

Sequence-analysis (Table 6, 7) had the result that changes in the mode of representation (drawing and writing), reference to written or drawn objects, reference to the test-material and to the actual problem and recapitulation were possible reasons for solution-ideas. The last surprising statement (there was no correlation between recapitulations and problem-score) could be understood by taking into account mistakes (see below).

So the heuristic value of perception is underlined with respect to creative production.

Unfortunately no systematical connection between impasses and other variables could be found.

Hereby the conjecture is supported that success seems to be more dependent with systematics than failure (cf. 4.1).

4.3 The significance of the modes of representation within the problem solving process

The contrast established by the low correlation between problem-score and strong systematic connections between solution-ideas and perception (cf. 3.4) may be explained by taking into account that different modes of representation were not only followed by solution-ideas but also by mistakes more often than it could be expected by chance (cf. Table 6, 7, 10). The same statement holds with respect to recapitulations (priority of recapitulations before changes in the mode of representation in sequence-analysis).

The attempt to determine possible reasons for changes in the mode of representation (e. g. by use of Table 8 and careful analysis of all protocols "by hand") resulted in the following list:
There were changes in the mode of representation because of

(i) systematic reasons (e. g. some subjects made nearly always a diagram after the first reading of the problem),

(ii) critical situations (e. g. impasses),

(iii) recording a (subjective) solution-idea,

(iv) writing down the result of a chain of reasoning.

4.4 Problem solving styles

Interpretation of Table 9 might result in the following list of properties of the iconoic and enactive style respectively (Table 11):

Table 11

Table 11

Comparison of properties of iconic and enactive problem solvers

iconic style (drawing)	enactive style (laying)
reflexive (slow psychic tempo, few mistakes) esp. mistakes due to incomplete reasons and external experience	impulsive (high psychic tempo, many mistakes) esp. mistakes due to functional fixtness, the colloquial language and confusion
seldom correction of mistakes	often correction of mistakes, esp. due to confusion
less communicative (cf. esp. verbal factor)	more communicative (cf. esp. verbal factor) esp. more often interpretation of own statements or of the text, planning ahead, judging of reasoning
more introverted and text-oriented (esp. intention to understand the problem, reading expressively, recall text)	
more systematics (esp. concerning drawing, cf. e. g. Table 8)	More mental flexibility. Esp. more often "sudden insight" (change of centration), logical negation (Theorem 2: proof by contradiction) and additional assumption
less reference to writing	more reference to writing
older	younger
somewhat more successful	less successful

4.5 Error-analysis

The difficulty to recognize existence problems may be explained e. g. by the fact that the subjects rather often incorporated the proposition into the supposition of a theorem (esp. Theorem 1, 2). Furthermore subjects frequently generated the existence of (geometrical) objects by means of infering the validity of a statement A from a clause "If A, than B" (theorems and axioms as well). These important mistakes were superposed (and explainable)by the fact that very often a mere drawing seems to make further logical arguments superfluous.

Analyzing correction of mistakes shows that all corrections of logical mistakes occurred in Theorem 2 and 3 only. Therefore it may be assumed that the consciousness for illegal thinking develops with growing experience.

5. Implication for future research

Some prospects can be made to future research. Possible implications can be subdivided into two groups:

a) Conclusion, completion and supplementation of the present study, particularly testing the stated hypotheses.

b) Some speculations (hypotheses) about possible instruction in solving problems similar to those used in the study.

ad a)

- The very detailed standardized protocols of this study could not only be translated into the language of Dörner (cf. 3.2), but also into the language of Wertheimer (1945), Duncker (1945), Polya (1957), Kilpatrick (1067) or Kantowski(1974). So it might be possible to "factorize" (condense) the set of data of this study with respect to different points of view and goals, rendering comparison more easily.
- Another possibility would be the analysis of the communication-structures between subject and interviewer corresponding to the fundamental idea that learning and teaching of mathematics is a communication-process (cf. Bauersfeld (1978)). In the present study main stress was laid on the problem solving process in the subject only. Thus the videotapes have only been translated as far as standardized protocols in which the interviewer gave no aid to the subject.
- As this study was an exploratory one, it must be tested, of course, how far the results presented in 3. and 4. can be generalized. Therefore studies must be carried out with a larger class of subjects and problems.
- The hypotheses concerning the iconic and enactive problem solving style should now be tested by a longitudinal study at primary-school level (cf. Rinkens (1978)). The development of process-variables

should be analyzed in this way in more detail.

- These studies should be carried out in even more connection and cooperation with students and teachers to make the results and the language of their description even more fit to the self-understanding of the applicator of problem solving research (cf. the action-research-approach, e. g. Moser (1975)).
- A strong need seems to be not only towards more praxis-orientation of problem solving research (see above), but also towards more elaborated theoretical guidelines (cf. 1.). A furthergoing attempt is presented in Zimmermann (1979). The import of norms underlying all research in behavioral sciences (like mathematics education), is outlined and explained in that study.
- A further study is carried out, oriented to the information-processing-approach by which a theoretical model about mathematical problem solving should be developed. Ideas of "fuzzy logic" (cf. Le Faivre (1974)), "problem-logic" (Kolmogorov (1932)) and "time-logic" (Fischer (1976)) are connected in this model.The quality of this model was tested by computer-simulation. The main results of section 3.2 (transition-frequencies) could be reproduced in this way.

ad b)

If it were possible to confirm the hypotheses of this study , some speculations could be made in the direction of mathematical instructions (provided similar group of students and similar problem-class):

- Students should solve problems (to prove) in a systematic way, e. g. with the aid of Lüer's (1973) fundamental rhythm.
- Students should learn to be always aware, what they are doing or going to do. They should express their intentions, orient themselves to the goal and should plan ahead. Training of process-awareness seems to be useful (cf. Kantowski (1977)).

- Students should be encouraged to change the mode of representation in suitable situations and recapitulate own statements; so they may gain more (objective) solution-ideas. These activities should, however, be carried out critically and carefully to avoid mistakes. Esp. awareness of existence-problems should be trained. Perhaps education-programs which train the reasonable use of diagrams and enactive representation must be developed.
- Enactive and iconic types of problem solvers could be taught in different ways. The enactive ones should become more critical of their own thinking. They should orient themselves more to the written text than to their own statements. So there might be fewer mistakes. Some more systematics seem to be useful for them, too. Students of the iconic type should be more trained in the useful application of diagrams, to become even more successful.

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